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# Generic composition of boosts: an elementary derivation of the Wigner rotation

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**Abstract.** Because of its apparent complexity, the discussion of Wigner rotation is usually reduced to the study of Thomas precession, which is too specific a case to allow a deep understanding of boost composition. However, by using simple arguments and linear algebra, the result for the Wigner rotation is obtained straightforwardly, leading to a formula written in a manageable form. The result is exemplified in the context of the aberration of light.

## 1. Introduction

One of the most puzzling phenomena in special relativity is the composition of boosts. When one contemplates the form of an arbitrary boost [1], it becomes clear that the expression for the composition of two generic boosts will be very complicated. As is known, the composition of boosts does not result in a (different) boost but in a Lorentz transformation involving rotation (the *Wigner rotation* [2]), Thomas precession being the example normally worked out in textbooks [1,3–5]. In this example, one is composing two boosts along mutually perpendicular directions; for small velocities a second-order approximation allows one to get a result that is appropriate to understand the precession of the spin of an electron inside an atom.

Of course, the composition of two arbitrary boosts has also been studied in the literature [6–8], but generally the treatments are too involved to capture the Wigner rotation easily. Sometimes the papers are aimed at the understanding of certain properties of the Lorentz group, instead of looking for a straightforward way to get the Wigner rotation, leaving in the reader the impression that this topic is complicated, and cannot be comprehended without an involved analysis. Moreover, the expressions are often difficult to use in practice, and the concepts are frequently hidden behind the abundance of mathematics. The composition of boosts and the Wigner rotation are therefore virtually absent from textbooks (save for the very specific case of Thomas precession). One is then left with the impression that the subject is subtle and difficult. Of course this is true, but not to the point of preventing its treatment with simple mathematical tools.

In this paper the aim will be different. Our prime interest is in the Wigner rotation; we choose the composition of boosts as a specific issue, because some characteristics of boosts are

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highlighted particularly well, the power of linear analysis is demonstrated at its best, and, of course, because it is interesting in itself. The mathematical tool that we will use is simple linear algebra. After all, boosts are linear transformations. However, the key point is that boosts are symmetric linear transformations. This simple property will allow us to effortlessly compute the Wigner rotation (see equation (8) below). Moreover, the understanding of the reason that makes the boosts symmetric will reveal some simple, basic facts that are often passed over in textbook treatments. A second goal of this paper is to present simple formulae for computing the Wigner rotation. Their simplicity does not reside in their explicit form; the final result will always be messy. However, we want to give equations that are operationally simple in order that the computation of the Wigner rotation should be a simple ‘plug-and-play’ procedure.

## 2. Composition of boosts

We will start by considering the composition of two boosts along mutually perpendicular directions. Before embarking upon calculation, one should be sure about what one is looking for: one is wondering whether the composition is equivalent to a single boost or not. There are various ways of understanding this topic, depending to a large degree on the particular expertise and taste of the reader. For the moment we will content ourselves with a mathematical explanation. In section 3, we will clarify the meaning of the Wigner rotation by a physical example concerning the aberration of light.

One could give an answer to the question by starting from the fact that boosts are represented by symmetric matrices. On the one hand one knows that a boost $B_x$ along the $x$-axis is actually represented by a symmetric matrix, and on the other hand one could get a generic boost by performing an arbitrary spatial rotation: $B_x \longrightarrow \mathcal{R} B_x \mathcal{R}^{-1}$. Since the rotations are orthogonal matrices, then a boost along an arbitrary direction is also represented by a symmetric matrix $B = \mathcal{R} B_x \mathcal{R}^{\mathrm{T}}$ $(B^{\mathrm{T}} = B)$, whose form can be found in the literature [1]. This symmetry can also be regarded as a reflection of the fact that boosts leave four independent directions in spacetime invariant: namely, (i) they do not modify the light-cones; on the light-cone there are two independent directions, belonging to light-rays travelling back and forth along the boost direction, that remain invariant (see appendix A); (ii) in addition, the spacelike directions that are perpendicular to the boost direction are also left unchanged (a further two independent directions). Then, boosts have four independent real eigen(four)-vectors, and their representative matrices must be symmetric (i.e. diagonalizable). In contrast, a (spatial) rotation changes the directions belonging to the plane where it is performed.

Since the product of matrices representing boosts is non-symmetric (unless both boosts are parallel), then one can answer that the composition of two boosts is not, in general, equivalent to a single boost. So we are compelled to analyse the result of the composition of two boosts as being equivalent to the composition of a boost and a rotation. Again the symmetry of the boosts will allow us to identify the rotation in the result.

### *2.1. Composition of mutually perpendicular boosts*

Let there be two boost matrices along the $x$ and $y$ directions:

$$B_{(x)} = \begin{pmatrix} \gamma_1 & -\gamma_1\beta_1 & 0 & 0 \\ -\gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

$$B_{(y)} = \begin{pmatrix} \gamma_2 & 0 & -\gamma_2\beta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

The product of these two matrices yields

$$B_{(y)}B_{(x)} = \begin{pmatrix} \gamma_2 & 0 & -\gamma_2\beta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_2\beta_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\gamma_1\beta_1 & 0 & 0 \\ -\gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_2\gamma_1 & -\gamma_2\gamma_1\beta_1 & -\gamma_2\beta_2 & 0 \\ -\gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ -\gamma_2\gamma_1\beta_2 & \gamma_2\gamma_1\beta_2\beta_1 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

which is non-symmetric, as anticipated. Note that if one wants to speak about inertial systems, there are three of them here: the initial system from which $\beta_1$ is defined, the second which is the result of applying the first boost and from which $\beta_2$ is measured, and the final one obtained as a result of making the second boost. These systems are all taken with their spatial axis parallel to the previous one. These considerations are not important in working out the computations, but are crucial when one wants to interpret them physically. So, we will write equation (3) as the product of a boost $B_f$ and a rotation $R$†:

$$B_{(y)}B_{(x)} = RB_f \tag{4}$$

where

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_W & \sin\theta_W & 0 \\ 0 & -\sin\theta_W & \cos\theta_W & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Therefore

$$B_f = R^{-1}B_{(y)}B_{(x)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_W & -\sin\theta_W & 0 \\ 0 & \sin\theta_W & \cos\theta_W & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_2\gamma_1 & -\gamma_2\gamma_1\beta_1 & -\gamma_2\beta_2 & 0 \\ -\gamma_1\beta_1 & \gamma_1 & 0 & 0 \\ -\gamma_2\gamma_1\beta_2 & \gamma_2\gamma_1\beta_2\beta_1 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_2\gamma_1 & -\gamma_2\gamma_1\beta_1 & -\gamma_2\beta_2 & 0 \\ (-\gamma_1\beta_1\cos\theta_W + \gamma_2\gamma_1\beta_2\sin\theta_W) & (\gamma_1\cos\theta_W - \gamma_2\gamma_1\beta_2\beta_1\sin\theta_W) & -\gamma_2\sin\theta_W & 0 \\ (-\gamma_1\beta_1\sin\theta_W - \gamma_2\gamma_1\beta_2\cos\theta_W) & (\gamma_1\sin\theta_W + \gamma_2\gamma_1\beta_2\beta_1\cos\theta_W) & \gamma_2\cos\theta_W & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

The angle $\theta_W$ can be obtained by demanding the symmetry of the matrix $B_f$:

$$-\gamma_2\sin\theta_W = \gamma_1\sin\theta_W + \gamma_2\gamma_1\beta_2\beta_1\cos\theta_W \tag{7}$$

i.e.

$$\tan\theta_W = -\frac{\gamma_2\gamma_1\beta_2\beta_1}{\gamma_2 + \gamma_1} \tag{8}$$

or

$$\sin\theta_W = -\frac{\gamma_2\gamma_1\beta_2\beta_1}{\gamma_2\gamma_1 + 1} \qquad \cos\theta_W = \frac{\gamma_2 + \gamma_1}{\gamma_2\gamma_1 + 1}. \tag{9}$$

† One could also opt for $B'_f R$. The argument is the same; note also that $RB_f = B'_f R$ implies $B_f = R^T B'_f R$.

By replacing these values, one finds that the boost $B_f$ is

$$B_f = \begin{pmatrix} \gamma_2\gamma_1 & -\gamma_2\gamma_1\beta_1 & -\gamma_2\beta_2 & 0 \\ -\gamma_2\gamma_1\beta_1 & \left(1+\dfrac{\gamma_2^2\gamma_1^2\beta_1^2}{\gamma_2\gamma_1+1}\right) & \dfrac{\gamma_2^2\gamma_1\beta_2\beta_1}{\gamma_2\gamma_1+1} & 0 \\ -\gamma_2\beta_2 & \dfrac{\gamma_2^2\gamma_1\beta_2\beta_1}{\gamma_2\gamma_1+1} & \dfrac{\gamma_2(\gamma_2+\gamma_1)}{\gamma_2\gamma_1+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (10)$$

which is a boost along some direction in the $x$–$y$ plane. In order to find this direction, we will look for the direction in the $x$–$y$ plane that is left invariant by the boost $B_f$, i.e. the direction that is orthogonal to the direction of the boost. Since the vectors that are orthogonal to the direction of the boost do not suffer changes (either in direction or magnitude), one can write $B_f w = w$ for such a four-vector, or

$$\begin{pmatrix} \gamma_2\gamma_1 & -\gamma_2\gamma_1\beta_1 & -\gamma_2\beta_2 & 0 \\ -\gamma_2\gamma_1\beta_1 & \left(1+\dfrac{\gamma_2^2\gamma_1^2\beta_1^2}{\gamma_2\gamma_1+1}\right) & \dfrac{\gamma_2^2\gamma_1\beta_2\beta_1}{\gamma_2\gamma_1+1} & 0 \\ -\gamma_2\beta_2 & \dfrac{\gamma_2^2\gamma_1\beta_2\beta_1}{\gamma_2\gamma_1+1} & \dfrac{\gamma_2(\gamma_2+\gamma_1)}{\gamma_2\gamma_1+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ w^x \\ w^y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ w^x \\ w^y \\ 0 \end{pmatrix}. \qquad (11)$$

As a consequence $\gamma_1\beta_1 w^x + \beta_2 w^y = 0$, which can be read by saying that the vector $w^x\hat{x} + w^y\hat{y}$, in the $x$–$y$ plane, is orthogonal to the vector $\gamma_1\beta_1\hat{x} + \beta_2\hat{y}$. Thus this last vector is in the direction of the boost $B_f$. In order to identify the velocity of the boost $B_f$, one could consider the displacement four-vector between two events that happen at the same place in the original coordinate system: $\Delta = (\Delta\tau, 0, 0, 0)$, $\Delta\tau$ being the proper time. Since $\Delta \to B_f\Delta$, in the boosted coordinate system the time interval between the events is $\gamma_2\gamma_1\Delta\tau$. From the known relation between proper time and coordinate time, one obtains the result that the gamma factor (in other words, the velocity) of the boost $B_f$ is $\gamma_f = \gamma_2\gamma_1$. Then $\beta_f^2 = 1 - \gamma_f^{-2} = 1 - \gamma_2^{-2}\gamma_1^{-2} = 1 - (1-\beta_2^2)(1-\beta_1^2) = \beta_1^2 + \gamma_1^{-2}\beta_2^2$. This result, together with the direction of the boost, completes our understanding of the transformation $B_f$†.

In summary, the composition of a boost along the $x$ axis with velocity $\beta_1$ followed by a boost along the $y$ axis with velocity $\beta_2$ is equivalent to a single boost with velocity $\boldsymbol{\beta}_f = \beta_1\hat{x} + \gamma_1^{-1}\beta_2\hat{y}$ (the relativistic composition of velocities), followed by a rotation in the $x$–$y$ plane by an angle $\theta_W = -\arctan\left(\gamma_2\gamma_1\beta_2\beta_1/(\gamma_2+\gamma_1)\right)$, i.e.

$$B_{(y)}(\beta_2)\; B_{(x)}(\beta_1) = R(\theta_W)\; B_f \qquad (12)$$

where

$$\boldsymbol{\beta}_f = \beta_1\hat{x} + \gamma_1^{-1}\beta_2\hat{y} \qquad (13)$$

and as before

$$\tan\theta_W = -\frac{\gamma_2\gamma_1\beta_2\beta_1}{\gamma_2+\gamma_1}. \qquad (8)$$

As a preparation for the next section, note that we can read (12) backward to note that any boost $B$ in the $x$–$y$ plane can be decomposed into two mutually perpendicular boosts followed by a rotation:

$$B = R^{-1}\; B_{(y)}\; B_{(x)}. \qquad (14)$$

† Alternatively, the velocity of a boost $B(\beta)$ can be straightforwardly read from the first file of its matrix. Indeed, in order that the time transformation adopts a form manifestly invariant under spatial rotations, $ct' = \gamma(ct - \boldsymbol{\beta}\cdot\boldsymbol{r})$, the first file must be $(\gamma, -\gamma\boldsymbol{\beta})$.

### *2.2. Composition of arbitrary boosts*

Equipped with the previous understanding of the composition of two perpendicular boosts, let us tackle the general case. A generic composition of boosts can be seen as the composition of a boost $B_{(a)}$ of velocity $\boldsymbol{\beta}_a$, and a second boost $B$ of velocity $\boldsymbol{\beta} = \boldsymbol{\beta}_\parallel + \boldsymbol{\beta}_\perp$, where $\parallel$ and $\perp$ mean the parallel and perpendicular directions with respect to the first boost $\boldsymbol{\beta}_a$. Since the Wigner rotation is a geometric result (it only depends on the velocities of the boosts and the angle between them), one is free to choose the $x$–$y$ plane as the plane defined by both velocities, the $x$ axis as the direction $\parallel$, and the $y$ axis as the direction $\perp$. Although a generic composition of boosts could demand formidable algebraic manipulations, we will be able to get the result by using only the results of the previous section. The key to attaining our goal will be the decomposition (14). In fact, the main difficulty comes from the fact that the second boost has components $\hat{x}$ and $\hat{y}$. Our first step will consist in rewriting the second boost $B$ as a composition of a boost along $\hat{x}$ and another boost along $\hat{y}$. This was done formally at the end of the preceding section. We can thus use equation (14) to regard the second boost $B(\boldsymbol{\beta} = \beta_\parallel \hat{x} + \beta_\perp \hat{y})$ as a product of a rotation and two mutually perpendicular boosts, i.e.

$$B(\boldsymbol{\beta}) = R^{-1}(\phi)\ B_{(y)}(\beta_2 \hat{y})\ B_{(x)}(\beta_\parallel \hat{x}) \tag{15}$$

where

$$\beta_2 = \gamma_\parallel \beta_\perp \tag{16}$$

in order that the relativistic composition of the velocities $\beta_\parallel \hat{x}$ and $\beta_2 \hat{y}$ gives back $\boldsymbol{\beta} = \beta_\parallel \hat{x} + \beta_\perp \hat{y}$. Then $\gamma_2 = \gamma \gamma_\parallel^{-1}$, with $\gamma = \gamma(\beta)$, and

$$\tan\phi = -\frac{\gamma_2 \gamma_\parallel \beta_2 \beta_\parallel}{\gamma_2 + \gamma_\parallel} = -\frac{\gamma \gamma_\parallel \beta_\perp \beta_\parallel}{\gamma \gamma_\parallel^{-1} + \gamma_\parallel}. \tag{17}$$

At first glance it would seem to the reader that we are going backward, decomposing the boosts instead of composing them. The advantage of doing this will become clear in a few lines. We can now turn to the composition of $B(\boldsymbol{\beta})$ and $B_{(a)}(\beta_a \hat{x})$:

$$\begin{aligned} B(\boldsymbol{\beta})\ B_{(a)}(\beta_a \hat{x}) &= R^{-1}(\phi) B_{(y)}(\beta_2 \hat{y})\ B_{(x)}(\beta_\parallel \hat{x})\ B_{(a)}(\beta_a \hat{x}) \\ &= R^{-1}(\phi) B_{(y)}(\beta_2 \hat{y})\ B_{(x)}(\beta_1 \hat{x}) \end{aligned} \tag{18}$$

where

$$\beta_1 = \frac{\beta_\parallel + \beta_a}{1 + \beta_\parallel \beta_a} \tag{19}$$

denotes the velocity corresponding to the composition of two parallel boosts (then $\gamma_1 = \gamma_\parallel \gamma_a (1 + \beta_\parallel \beta_a)$). Note that we combined the two consecutive boosts in the $\hat{x}$ direction using the well known velocity addition formula. In this way one falls back to the composition of the two remaining mutually perpendicular boosts. At this point, let us recall our objective: we want to regard the composition $B(\boldsymbol{\beta})\ B_{(a)}(\beta_a \hat{x})$ as the product of a rotation $R(\theta_{\mathrm{W}})$ in the $x$–$y$ plane and a boost $B_f$. Then

$$R(\theta_{\mathrm{W}}) B_f = B(\boldsymbol{\beta})\ B_{(a)}(\beta_a \hat{x}) = R^{-1}(\phi)\ B_{(y)}(\beta_2 \hat{y})\ B_{(x)}(\beta_1 \hat{x}) \tag{20}$$

which means

$$R(\theta_{\mathrm{W}} + \phi) B_f = B_{(y)}(\beta_2 \hat{y})\ B_{(x)}(\beta_1 \hat{x}). \tag{21}$$

The good news is that we have already solved this expression in the previous section. The matrix $B_f$ is that of (10) with the velocities of (16) and (19). As shown there, $B_f$ is a boost whose velocity $\boldsymbol{\beta}_f$ comes from the relativistic composition of the velocities $\beta_1 \hat{x}$ and $\beta_2 \hat{y}$:

$$\boldsymbol{\beta}_f = \beta_1 \hat{x} + \gamma_1^{-1} \beta_2 \hat{y} = \frac{\beta_\parallel + \beta_a}{1 + \beta_\parallel \beta_a} \hat{x} + \frac{\gamma_a^{-1} \beta_\perp}{1 + \beta_\parallel \beta_a} \hat{y} \tag{22}$$

i.e. $\boldsymbol{\beta}_f$ is the relativistic composition of $\boldsymbol{\beta}_a$ and $\boldsymbol{\beta}$. The angle $(\theta_W + \phi)$ in (21) must satisfy (8):

$$\tan(\theta_W + \phi) = -\frac{\gamma_2\gamma_1\beta_2\beta_1}{\gamma_2 + \gamma_1} = -\frac{\beta_\perp(\beta_\parallel + \beta_a)}{\gamma_\parallel^{-2}\gamma_a^{-1} + \gamma^{-1}(1 + \beta_\parallel\beta_a)} \equiv \zeta. \tag{23}$$

Since $\tan(\theta_W + \phi) = (\tan\theta_W + \tan\phi)/(1 - \tan\theta_W \tan\phi)$, one concludes that the Wigner rotation for the composition $B(\boldsymbol{\beta} = \beta_\parallel\hat{x} + \beta_\perp\hat{y})\ B_{(a)}(\boldsymbol{\beta}_a = \beta_a\hat{x})$ is a rotation in the spatial plane defined by the directions of both boosts, whose angle $\theta_W$ is given by

$$\tan\theta_W = \frac{\zeta - \tan\phi}{1 + \zeta\tan\phi}. \tag{24}$$

Recall that $\parallel$ and $\perp$ in these equations mean the parallel and perpendicular directions with respect to the first boost $\boldsymbol{\beta}_a$, in the spatial plane defined by both boosts $\boldsymbol{\beta}_a$ and $\boldsymbol{\beta}$. The velocity $\boldsymbol{\beta} = \beta_\parallel\hat{x} + \beta_\perp\hat{y}$ is measured by an observer at rest in the system defined by the first boost $\boldsymbol{\beta}_a$. Note that $\zeta$ and $\phi$ are readily obtained from the data, namely $\beta_a$, $\beta_{||}$ and $\beta_\perp$ via equations (23) and (17).

## 3. Aberration of light

We will show an application of Wigner rotation in the context of the aberration of light (i.e. the change in the direction of propagation of a light-ray produced by a boost). For simplicity we shall work with two mutually perpendicular boosts. Let us choose the $x$ axis to coincide with the propagation direction of the light-ray. A first boost $B_{(x)}(\beta_1)$ leaves the propagation direction invariant, while a second boost $B_{(y)}(\beta_2)$ changes that direction according to the law of the aberration of zenithal starlight:

$$\delta_c = \arccos\gamma_2^{-1} \tag{25}$$

where $\delta_c$ is the angle between the $x$ direction in the original coordinate system (the light-ray) and the $x$ direction after the composition. This is *not* the aberration angle due to a boost with the relativistically composed velocity $\boldsymbol{\beta}_f = \beta_1\hat{x} + \gamma_1^{-1}\beta_2\hat{y}$. The Wigner rotation provides the difference between these two angles.

In fact, in appendix B the aberration angle for a boost with velocity $\boldsymbol{\beta}_f = \beta_1\hat{x} + \gamma_1^{-1}\beta_2\hat{y}$ has been computed; the result is

$$\delta = \arccos\left[\frac{\left(\beta_1^2 + \beta_2^2(1+\beta_1)(\gamma_2^{-1}\gamma_1^{-1} - \beta_1)\right)}{(\beta_1^2 + \gamma_1^{-2}\beta_2^2)}\right]. \tag{26}$$

The difference between equations (25) and (26) is due to the fact that the new $x$ direction in the two processes is not the same. So the boost associated with the relativistically composed velocity $\boldsymbol{\beta}_f$ must be completed with a rotation, in order to yield the aberration coming from the composition of boosts. The rotation angle $\delta - \delta_c$ is the Wigner angle (8). To make contact with our previous method, what we are saying is that in the first case

$$B_{(y)}(\beta_2)B_{(x)}(\beta_1)\begin{pmatrix} c \\ c \\ 0 \\ 0 \end{pmatrix} = \gamma_1\gamma_2(1-\beta_1)\begin{pmatrix} c \\ c\cos(\delta_c) \\ c\sin(\delta_c) \\ 0 \end{pmatrix}$$

while in the second case

$$R(\theta_W)B_f(\beta_1\hat{x} + \gamma_1^{-1}\beta_2\hat{y})\begin{pmatrix} c \\ c \\ 0 \\ 0 \end{pmatrix} = R(\theta_W)\gamma_1\gamma_2(1-\beta_1)\begin{pmatrix} c \\ c\cos(\delta) \\ c\sin(\delta) \\ 0 \end{pmatrix}$$

$$= \gamma_1\gamma_2(1-\beta_1)\begin{pmatrix} c \\ c\cos(\delta-\theta_W) \\ c\sin(\delta-\theta_W) \\ 0 \end{pmatrix}.$$

Since $B_{(y)}(\beta_2)B_{(x)}(\beta_1) = R(\theta_W)B_f(\beta_1\hat{x} + \gamma_1^{-1}\beta_2\hat{y})$, then $\delta - \theta_W = \delta_c$ as stated above. The multiplicative factor $\gamma_1\gamma_2(1-\beta_1)$ is the Doppler shift.

## 4. Conclusions

Our argument for working out the Wigner rotation can then be given in a nutshell as follows. First, a boost along the $x$ direction is manifestly symmetric. One can also understand this feature by noting that there are two null eigenvectors along the null cone (with eigenvalue equal to the Doppler shifts) and two trivial ones (along the $y$ and $z$ axes). Now, since a generic boost is obtained by a rotation of the axis and $R^{-1} = R^T$ (i.e. $R$ is orthogonal), the matrix representing a generic boost stays symmetric (or, equivalently, it will preserve its four eigenvectors with real eigenvalues). The symmetry allows us to easily compute the Wigner angle in the case of the composition of two perpendicular boosts. Now in the generic case, the problem can be cast in a form identical to the previous one, after carrying out a proper decomposition of the boosts into two mutually perpendicular directions. Thus the answer is written without any difficult algebraic computing.

Physically not intuitive due to the lack of any Galilean analogue, Wigner rotation has been relegated to some corner of knowledge. Although Wigner rotation is challenging both in terms of mathematical skill and physical intuition, its computation is nonetheless within the reach of elementary analysis and it is an instructive way to apprehend the subtlety inherent in the subject.

## Appendix A. Eigen-directions of a boost

We will show the two null eigen-directions of a boost explicitly. Let the boost be in the $\hat{x}$ direction; dropping the two invariant spatial directions $\hat{y}$ and $\hat{z}$, and working just in the $t$–$x$ plane, the orthogonal transformation required is

$$OB_x(\beta)O^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \gamma(1+\beta) & 0 \\ 0 & \gamma(1-\beta) \end{pmatrix}. \quad \text{(A1)}$$

The coordinate change is simply

$$u = \frac{1}{\sqrt{2}}(ct - x) \quad \text{(A2)}$$

$$v = \frac{1}{\sqrt{2}}(ct + x) \quad \text{(A3)}$$

which are the so-called null coordinates. The eigenvalues associated with the null directions are the relativistic Doppler shift factors (this is, of course, not a surprising result). This change of coordinates is not a Lorentz transformation, because it does not leave the Minkowski metric invariant:

$$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad \text{(A4)}$$

This is evident when we look at the transformation on a Minkowski diagram: it amounts to a rigid rotation of 45° in the anticlockwise sense in space–time instead of the famous 'scissor-like' picture of the Lorentz transformation. This can be traced to the fact that the proper Lorentz group is isomorphic to $O(1, 3)$ instead of $O(4)$. The matrix $O$ in (A1) belongs to the group $O(4)$.

## Appendix B. Computation of the aberration angle

To begin with, we will recall the aberration angle due to a boost $B_{(x)}(\beta)$. If the light-ray propagates in the direction $\hat{n} = (\cos\psi, \sin\psi, 0)$, the transformed direction $\hat{n}'$ is obtained by applying the usual Lorentz transformation to the velocity $\boldsymbol{u} = c\hat{n}$, which transforms to $\boldsymbol{u}' = c\hat{n}'$:

$$\hat{n}' = \left(\frac{\cos\psi - \beta}{1 - \beta\cos\psi}, \frac{\sin\psi}{\gamma(1 - \beta\cos\psi)}, 0\right). \tag{B1}$$

The aberration angle is

$$\cos\delta = \hat{n}\cdot\hat{n}' = \frac{1}{1 - \beta\cos\psi}\left[\cos\psi(\cos\psi - \beta) + \gamma^{-1}\sin^2\psi\right]. \tag{B2}$$

In getting this result, the $x$-axis was chosen in the direction of the boost because of practical reasons. But, of course, the aberration angle depends only on the norm of $\boldsymbol{\beta}$ and the angle $\psi$ between $\boldsymbol{\beta}$ and the light-ray.

Let us now study the problem proposed in the body of the text. Let there be a boost with velocity $\boldsymbol{\beta}_f = \beta_1\hat{x} + \gamma_1^{-1}\beta_2\hat{y}$, and a light-ray travelling along the $x$ axis. Then, using the substitutions

$$\cos\psi = \frac{\beta_1}{\beta_f} = \frac{\beta_1}{\sqrt{\beta_1^2 + \gamma_1^{-2}\beta_2^2}} \qquad \sin\psi = -\frac{\gamma_1^{-1}\beta_2}{\beta_f} = -\frac{\gamma_1^{-1}\beta_2}{\sqrt{\beta_1^2 + \gamma_1^{-2}\beta_2^2}}$$

in (B2) (the minus sign is due to the fact that the angle $\psi$ is measured in the anticlockwise sense from $\boldsymbol{\beta}_f$ to $\hat{n}$), after some algebra one obtains

$$\cos\delta = \frac{\beta_1^2 + \beta_2^2(1 + \beta_1)(\gamma_2^{-1}\gamma_1^{-1} - \beta_1)}{\beta_1^2 + \gamma_1^{-2}\beta_2^2} \tag{B3}$$

i.e. in the boosted system the angle between the light-ray (the $x$ direction in the original coordinate system) and the boost direction is $\psi' = \psi + \delta$.

The result (B3) can be compared with that corresponding to the boost composition $B_{(y)}(\beta_2)B_{(x)}(\beta_1)$. The first boost does not produce aberration, since it has the same direction as the light-ray. The second produces an aberration that is a particular case of (B3) with $\beta_1 = 0$:

$$\cos\delta_c = \gamma_2^{-1}. \tag{B4}$$

Of course, the same result is recovered from (B2) by replacing $\beta = \beta_2$ and $\psi = \pi/2$.

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